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On the Fourier transform and the spectral properties of the p-adic momentum and Schrödinger operators

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Abstract. A momentum representation for p-adic quantum mechanics is constructed. It uses a Fourier transform in the L_2-space with respect to the p-adic Lebesgue distribution and a study of its properties. The momentum representation is used to investigate the spectral properties of the p-adic momentum operator and to study the (p-adic valued) Schrödinger equation for the p-adic harmonic oscillator.

1. Introduction

In this paper we continue our investigations on spectral properties of quantum operators realized in a p-adic Hilbert space (see, for example [1–4] for the quantization in the p-adic Hilbert space and [5–8] for p-adic mathematical physics). In particular, we study a p-adic Hilbert space representation of the canonical commutation relation:

$$[\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}] = \mathrm{i}\boldsymbol{I}. \tag{1}$$

In [9] it was shown that one can realize the position and momentum operators $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ as bounded symmetric operators in the p-adic Hilbert space. Then, in [10] we studied the spectral properties of the position operator $\hat{\boldsymbol{x}}$ in the L_2-space with respect to the p-adic Gaussian distribution [1]. There is a large difference from the standard representation of the position operator in a complex (separable) Hilbert space. The spectrum of the p-adic position operator is concentrated in a ball of finite radius. The radius of this ball depends on the covariance b of the p-adic Gaussian distribution. We remark that, in the p-adic case, non-equivalent Gaussian distributions exist and, consequently, non-isomorphic L_2-spaces with respect to these distributions. We can interpret this choice of the covariance b, as a fixing of the exactness of the position measurement. This can be seen as a realization of an old idea in the discussion of foundations of quantum mechanics that one should take care of physical equipment in the discussion of the structure of the space of quantum states. In our framework the fixing of equipment implies the fixing of an exactness for measurement, expressed in terms of a p-adic representation.

In this paper we study the spectral properties of the p-adic momentum operator $\hat{\boldsymbol{p}}$. To do this, we construct the p-adic momentum representation using an analogue of the usual Fourier transform. From the physical point of view, the most interesting result is that, in general, the spaces of quantum states of position and momentum do not coincide.

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This is not so strange from the experimental point of view; in fact, different preparation procedures $\mathcal{E}$ (e.g. those corresponding to given values of the positions and the momentum respectively) may produce different quantum states. For example, if we prepare a quantum state ϕ using a preparation procedure $\mathcal{E}_{\text{pos}}$ based on a position filter (i.e. we can formally write $\phi = \phi(\mathcal{E}_{\text{pos}})$), then from the experimental point of view it is not evident that we may prepare the same state ϕ by using some preparation procedure $\mathcal{E}_{\text{mom}}$ based on a momentum filter. However, if the standard formalism of quantum mechanics is used, then applying the Fourier transform we can change the representation and (in general, only theoretically) realize each quantum state $\phi = \phi(\mathcal{E}_{\text{pos}})$ (the position representation) as the state $\phi = \phi(\mathcal{E}_{\text{mom}})$ (the momentum representation). More generally, if $\{e_n\}$ and $\{e'_n\}$ are two different orthonormal bases in the complex Hilbert space, then the operator of the change of coordinates from the base $\{e_n\}$ where $\phi = \sum_{n=0}^{\infty} \phi_n e_n$ to the base $\{e'_n\}$ where $\phi = \sum_{n=0}^{\infty} \phi'_n e'_n$ is a unitary operator. Therefore, a change of coordinates does not change the physical properties, i.e. physical properties of the state ϕ realized as $\phi = \sum_{n=0}^{\infty} \phi_n e_n$ coincide with physical properties of the state (which is denoted in the standard formalism of quantum mechanics by the same symbol ϕ) realized as $\phi = \sum_{n=0}^{\infty} \phi'_n e'_n$. Thus from the point of view of ordinary quantum mechanics there is no difference to prepare the quantum state ϕ as a mixture of states $\{e_n\}$ (for example, corresponding to the position preparation procedure $\mathcal{E}_{\text{pos}}$, say for a particle in a box) or as a mixture of states $\{e'_n\}$ (for example, corresponding to the momentum preparation procedure $\mathcal{E}_{\text{pos}}$ for a particle in a box). The situation is crucially changed in the formalism of p-adic quantum mechanics. In the p-adic Hilbert space an operator of a change of coordinates between two orthonormal bases, in general, does not preserve the norm. From the physical point of view this means that a change of representation may change physical properties (in fact, the exactness of a measurement) of the system. For this reason we really need to use the indexes 'pos' and 'mom' for quantum states to indicate which representation is used (how these quantum states were prepared). The equivalence of the different representation in the standard mathematical formalism of quantum mechanics was not accepted by everybody: for example, de Broglie [11, 12] gave a strong motivation for the position representation to be the only 'real physical representation', all other representations being then only consequences of the position representation. According to de Broglie, we can reconstruct all properties of the momentum representation on the basis of the position representation, but we cannot reconstruct the position representation on the basis of the momentum representation. The mathematical equivalence of these representations was considered by de Broglie as a mathematical trick. We do not, however, directly follow de Broglie: in our p-adic quantum mechanics the position and momentum representations appear in a symmetric way. Nevertheless, they give rise to different spaces of quantum states (see also Ballentine [13, 14] on the connection between a preparation procedure and the space of quantum states).

From the mathematical point of view, our theory of the p-adic Fourier transform (for p-adic valued functions on the p-adic space) also differs very much from the well known Fourier transform theory on p-adic space, see [15–17]. The most important difference is that our definition of the Fourier operator gives rise to a unitary isometric operator between two p-adic Hilbert spaces, whereas the standard Fourier transform for p-adic valued functions on the p-adic space [15–17] has a non-trivial kernel.

Using the momentum representation and the results of [10] on the spectral properties of the position operator in our approach to p-adic quantum mechanics, we shall give results on the spectrum of the p-adic momentum operator $\hat{\boldsymbol{p}}$.

In section 5 we shall study the Schrödinger equation for a harmonic oscillator: the statistical interpretation and an analogue of the Heisenberg uncertainty principle are discussed in details. In section 6 we shall present results on the p-adic description of finite exactness of measurements.

2. The p-adic L_2-space with respect to Gaussian and Lebesgue distributions

The field of p-adic numbers is denoted by the symbol $\mathbb{Q}_p$. As usual, we define p-adic balls $U_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leqslant r\}$, $a \in \mathbb{Q}_p, r > 0$. Set $U_r = U_r(0)$. Denote the field of complex p-adic numbers (the completion of the algebraic closure of $\mathbb{Q}_p$) by the symbol $\mathbb{C}_p$. We shall use this field to define the roots $b^{1/2}$ and $b^{1/4}$ for the elements $b \in \mathbb{Q}_p$.

As in our previous paper [10], we consider the case $p = 3 \bmod 4$.

Let $b \neq 0$ be a p-adic number, the p-adic Gaussian distribution (with covariance b) ν_b is defined as a $\mathbb{Q}_p$-linear functional (on the space of p-adic valued polynomials over $\mathbb{Q}_p$) by its moments

$$M_{2n} = \int x^{2n}\nu_b(\mathrm{d}x) = (2n)!\frac{b^n}{n!2^n} \qquad M_{2n+1} = \int x^{2n+1}\nu_b(\mathrm{d}x) = 0 \qquad n \in \mathbb{N}_0$$

with $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$. Then this integral is extended by linearity and continuity to some classes of analytic functions.

Formally (by analogy with the real case), we can write ν_b as

$$\nu_b(\mathrm{d}x) = \mathrm{e}^{-\frac{x^2}{2b}}\,\mathrm{d}x \tag{2}$$

(with dx being a 'suitably normalized' Lebesgue measure).

However, this is only a symbolic expression. It does not have a strict mathematical sense, because there is no p-adic valued analogue of the Lebesgue measure. Moreover, further we shall introduce (see (21) below) a p-adic Lebesgue distribution on the basis of the Gaussian one.

Let us introduce the analogue of Hermite polynomials on $\mathbb{Q}_p$ corresponding to the p-adic Gaussian distribution with the covariance b:

$$H_{n,b}(x) = \frac{n!}{b^n}\sum_{k=0}^{[n/2]}\frac{(-1)^k x^{n-2k}b^k}{k!(n-2k)!2^k} \qquad x \in \mathbb{Q}_p. \tag{3}$$

These polynomials are orthogonal with respect to the p-adic Gaussian measure ν_b. We shall also use the following representation for these Hermitian polynomials:

$$H_{n,b}(x) = (-1)^n\mathrm{e}^{x^2/2b}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2/2b}. \tag{4}$$

This representation holds on any ball of sufficiently small radius with its centre in zero (see [9] for details).

Let $p = 3,\ \bmod 4$, then the equation $x^2+1 = 0$ has no solutions in $\mathbb{Q}_p$. We can consider the quadratic extension $\mathbb{Q}_p(\mathrm{i})$ with $\mathrm{i} = \sqrt{-1}$ of $\mathbb{Q}_p$. In analogy with complex numbers we set $z = x + \mathrm{i}y$, $x, y \in \mathbb{Q}_p$, $\mathrm{i} = \sqrt{-1}$, and $\bar{z} = x - \mathrm{i}y$. The valuation on $\mathbb{Q}_p(\mathrm{i})$ is also denoted by $|\cdot|_p$, $|z|_p = \sqrt{||z|^2|_p}$, where $|z|^2 = z\bar{z} = x^2 + y^2$. We remark that $|z|^2$ assumes its values in the field $\mathbb{Q}_p$, whereas $|z|_p$ assumes its values in the field of real numbers.

In previous papers [9, 10] we gave the following definition of the L_2 space with respect to the p-adic Gaussian distribution. The space $L_2(\mathbb{Q}_p, \nu_b)$ of p-adic valued functions of a p-adic argument is the set

$$\left\{f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x),\ f_n \in \mathbb{Q}_p(\mathrm{i}) : \text{the series } \sum_{n=0}^{\infty}|f_n|^2 n!/b^n \text{ converges in } \mathbb{Q}_p\right\}.$$

Remark. If $p \neq 3$, mod 4, then $\mathrm{i} \in \mathbb{Q}_p$. Thus no quadratic extension with respect to i is possible. The realization of quantum mechanical objects then becomes more complicated.

The requirement as above for functions to belong to $L_2(\mathbb{Q}_p, \nu_b)$ is equivalent to the requirement that

$$\sigma_{n,b}^2(f) \equiv \frac{|f_n|_p^2 |n!|_p}{|b|_p^n} \to 0 \qquad \text{when } n \to \infty. \tag{5}$$

The norm and the inner product in $L_2(\mathbb{Q}_p, \nu_b)$ are defined by the equalities:

$$\|f\| = \max_n \sigma_{n,b}(f) \tag{6}$$

respectively

$$(f, g) = \int f(x)\overline{g(x)}\nu_b(\mathrm{d}x). \tag{7}$$

If we choose a different parameter b, we obtain, in general, a different $L_2(\mathbb{Q}_p, \nu_b)$ space; the next theorem investigates the relation between these L_2 spaces. First, we give the following lemma.

Lemma 2.1. For any $b, c \in \mathbb{Q}_p$, $b \neq 0$, we have:

$$\int H_{s,b}(x)\nu_c(\mathrm{d}x) = \frac{(2l)!(c-b)^l}{l!2^l b^{2l}} \tag{8}$$

if s is even and we have set $s = 2l$; the integral is zero if s is odd; the integral is equal 1 if $b = c$ and $s = 0$.

Proof. First, we remark that we can restrict our considerations to the case $b \neq c$, because if $b = c$ and $s \neq 0$, then both sides of lemma 2.1 are equal to zero (it is also evident that if $b = c$ and $s = 0$, then the integral is equal to 1). We have

$$\begin{aligned}
\int H_{s,b}(x)\nu_c(\mathrm{d}x) &= \int (-1)^s \mathrm{e}^{\frac{x^2}{2b}} \left(\frac{\mathrm{d}^s}{\mathrm{d}x^s}\mathrm{e}^{-\frac{x^2}{2b}}\right)\nu_c(\mathrm{d}x) \\
&= (-\mathrm{i}/b)^s \left(\mathrm{e}^{\frac{-y^2}{2b}} \int \mathrm{e}^{\frac{x^2}{2b}} \frac{\mathrm{d}^s}{\mathrm{d}y^s}\mathrm{e}^{-\frac{(x-\mathrm{i}yb)^2}{2b}}\nu_c(\mathrm{d}x)\right)\Big|_{y=0} \\
&= (-\mathrm{i}/b)^s \left(\mathrm{e}^{\frac{-y^2}{2b}} \frac{\mathrm{d}^s}{\mathrm{d}y^s}\mathrm{e}^{\frac{y^2 b}{2}} \int \mathrm{e}^{\mathrm{i}xy}\nu_c(\mathrm{d}x)\right)\Big|_{y=0} \\
&= (\mathrm{i}/b)^s \mathrm{e}^{-\frac{y^2 c}{2}} \mathrm{e}^{\frac{y^2}{2\eta}} \frac{\mathrm{d}^s}{\mathrm{d}y^s}\mathrm{e}^{-\frac{y^2}{2\eta}}\Big|_{y=0} = (\mathrm{i}/b)^s \mathrm{e}^{-\frac{y^2 c}{2}} H_{s,\eta}(y)\Big|_{y=0} \\
&= (\mathrm{i}/b)^s H_{s,\eta}(0)
\end{aligned} \tag{9}$$

where $\eta = \frac{1}{c-b}$. Now, by using the expression

$$H_{s,\eta}(0) = \frac{(-1)^l(2l)!\eta^{-l}}{l!2^l}$$

we obtain the proof. □

Remark. The exponential function $\mathrm{e}^{\mathrm{i}xy}$ does not belong to the L_2-space (for any value of the parameter y). However, the integral $J(y) = \int \mathrm{e}^{\mathrm{i}xy}\nu_c(\mathrm{d}x)$ (the Fourier transform of the Gaussian distribution) is well defined for the sufficiently small $|y|_p$, because Gaussian integrals are defined for all functions which are analytic on the ball U_τ, $\tau > \theta_c = p^{1/2(1-p)}\sqrt{|c|_p}$, see [10].

Remark. In the above proof we have changed the order of integration and differentiation. In fact to justify this procedure, we have to prove that:

$$\int \frac{\mathrm{d}}{\mathrm{d}y} \mathrm{e}^{\frac{by^2}{2}+\mathrm{i}xy} \nu_c(\mathrm{d}x) = \frac{\mathrm{d}}{\mathrm{d}y} \int \mathrm{e}^{\frac{by^2}{2}+\mathrm{i}xy} \nu_c(\mathrm{d}x).$$

As all exponential series converge as geometric progressions for small $|y|_p$, we can reduce this problem to the evident equality for the monomials:

$$\int \frac{\mathrm{d}}{\mathrm{d}y} y^m x^k \nu_c(\mathrm{d}x) = \frac{\mathrm{d}}{\mathrm{d}y} \int y^m x^k \nu_c(\mathrm{d}x).$$

We shall use the equality

$$|n!|_p = p^{(n-S_n)/(1-p)} \tag{10}$$

where $n = \sum_j n_j p^j$, $n_j = 0, 1, \ldots, p-1$, is a p-adic expansion of the natural number n and $S_n = \sum_j n_j$ [18]. Moreover, we shall use the following lemma.

Lemma 2.2. $S_{j+k} \leqslant S_j + S_k$ [10].

Theorem 2.1. If the p-adic numbers b and c verify the relation $|c|_p \leqslant |b|_p$, then

$$L_2(\mathbb{Q}_p, \nu_b) \subseteq L_2(\mathbb{Q}_p, \nu_c). \tag{11}$$

Proof. Let $f \in L_2(\mathbb{Q}_p, \nu_b)$ and write

$$f(x) = \sum_n f_n H_{n,b}(x)$$

where $\sigma^2_{n,b}(f) \to 0$ when $n \to \infty$.

Now we can expand f in the form

$$f(x) = \sum_n \tilde{f}_n H_{n,c}(x) \tag{12}$$

where

$$\tilde{f}_n = \frac{c^n}{n!} \int f(x) H_{n,c}(x) \nu_c(\mathrm{d}x). \tag{13}$$

At this moment this is only a formal expansion. We shall prove that this series converges in $L_2(\mathbb{Q}_p, \nu_c)$.

Expression (13) can be rewritten in the following form by using standard properties of integration by parts of the Gaussian distribution [1, pp 40–1]:

$$\tilde{f}_n = \frac{c^n}{n!} \sum_{m=n}^{\infty} f_m \frac{m!}{(m-n)!\ b^n} \int H_{m-n,b}(x) \nu_c(\mathrm{d}x).$$

Now, by using lemma 2.1, we obtain

$$\tilde{f}_n = \frac{c^n}{b^n n!} \sum_{l=0}^{\infty} f_{n+2l} \frac{(n+2l)!(c-b)^l}{b^{2l} l! 2^l}$$

and we estimate

$$\begin{aligned} \sigma^2_{n,c}(f) &\leqslant \frac{|c|_p^{2n}}{|n!|_p |c|_p^n |b|_p^{2n}} \max_{l=0\ldots\infty} \left[\sigma_{n+2l,b}(f)^2 |(n+2l)!|_p \frac{|b|_p^{n+2l} |c-b|_p^{2l}}{|l!|_p^2 |4|_p^l |b|_p^{4l}} \right] \\ &= \left|\frac{c}{b}\right|^n \max_{l=0\ldots\infty} \left[\sigma_{n+2l,b}(f)^2 \left|\frac{c-b}{b}\right|^{2l} p^{\frac{S_{n+2l}-2S_l-S_n}{p-1}} \right]. \end{aligned} \tag{14}$$

By using lemma 2.2 and the fact that $f \in L_2(\mathbb{Q}_p, \nu_b)$, we obtain that $\sigma^2_{n,c}(f)$ approaches 0 if the following relations hold:

$$\left|\frac{c-b}{b}\right| \leqslant 1 \tag{15}$$

$$\left|\frac{c}{b}\right| \leqslant 1. \tag{16}$$

Since relation (15) is a consequence of (16), we obtain the proof. □

A consequence is the following.

Theorem 2.2. If the p-adic numbers c, b have the same p-adic norms, then the spaces $L_2(\mathbb{Q}_p, \nu_b)$ and $L_2(\mathbb{Q}_p, \nu_c)$ coincide. However, the canonical inclusion (12), (13) is not the isomorphism of p-adic Hilbert spaces.

There are difficulties in directly defining the Fourier transform of maps in $L_2(\mathbb{Q}_p, \nu_b)$, since the maps, which are p-adic analogues of the Hermite functions, are not elements of $L_2(\mathbb{Q}_p, \nu_b)$. In fact, they are defined by

$$\phi_{n,b}(x) = b^{-1/4} H_{n,b}(x) \mathrm{e}^{-\frac{x^2}{4b}}. \tag{17}$$

The power expansion $\mathrm{e}^{-\frac{x^2}{4b}}$ is pointwise convergent only for $|x|_p < \theta$ where the constant θ was already defined in [10]

$$\theta \equiv \theta_b = p^{\frac{1}{2(1-p)}} \sqrt{|b|_p}. \tag{18}$$

In the following, we shall consider more than one covariance, and we shall add a suffix to θ to clarify the parameter on which it is depending on.

We remark that by computing the Hermitian coefficients of $\mathrm{e}^{-\frac{x^2}{4b}}$, we can easily show that $\mathrm{e}^{-\frac{x^2}{4b}}$ does not belong to $L_2(\mathbb{Q}_p, \nu_b)$.

For this reason we shall consider a different functional space, which we shall show to be isomorphic to $L_2(\mathbb{Q}_p, \nu_b)$.

Definition 2.1. $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ is the set

$$\left\{ f(x) = \sum_{n=0}^{\infty} f_n \phi_{n,b}(x),\ f_n \in \mathbb{Q}_p(\mathrm{i}) : \text{the series } \sum_{n=0}^{\infty} |f_n|^2 n!/b^n \text{ converges in } \mathbb{Q}_p \right\}.$$

Remark. The maps f are defined only for $|x|_p < \theta$.

On $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ we define the norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$. If $f \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ we define $\|f\|^2 = \max_n \sigma^2_{n,b}(f)$. Furthermore, if $u, v \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$, we define

$$(u, v) = \sum u_n \overline{v_n} \frac{n!}{b^n}. \tag{19}$$

In the next theorem we state that the space $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ is isomorphic to $L_2(\mathbb{Q}_p, \nu_b)$. If $f \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$, we set $f(x) = \sum f_n \phi_{n,b}(x)$ and define $U(f)(x) = \sum f_n H_{n,b}(x) \in L_2(\mathbb{Q}_p, \nu_b)$.

Theorem 2.3. The map U is an isomorphism (i.e. a unitary and isometric map) between the two p-adic Hilbert spaces, $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ and $L_2(\mathbb{Q}_p, \nu_b)$.

The reader can easily show that U preserves the norm and the inner product:

$$(f, g) = (U^{-1}(f), U^{-1}(g)). \tag{20}$$

By using our Gaussian distribution ν_b, we can define a Lebesgue integral for p-adic valued functions f of the form $f(x) = \phi(x)\mathrm{e}^{\frac{-x^2}{2b}}/\sqrt{b}$, $|x|_p < \theta$, where $\phi(x)$ is integrable with respect to the Gaussian distribution†:

$$\int f(x)\,\mathrm{d}x = \int \phi(x)\nu_b(\mathrm{d}x). \tag{21}$$

By using this integral, the scalar product just defined admits the following integral representation:

$$(u, v) = \int u(x)\overline{v(x)}\,\mathrm{d}x. \tag{22}$$

Finally, by using theorems 2.2 and 2.3, we easily obtain the following.

Corollary 2.1. If the p-adic numbers b, c verify $|b|_p = |c|_p$, then the spaces $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ and $L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$ are isomorphic as Banach spaces.

3. Fourier transform of L_2 maps

Recalling the role played by Hermite functions in the classical harmonic analysis, we define the Fourier transform as a map of a p-adic L_2-space into another p-adic L_2-space, as follows.

Definition 3.1. The continuous linear map $\mathcal{F} : L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x) \to L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$ for $c = \frac{1}{4b}$ defined as

$$\mathcal{F}(\phi_{n,b}(x)) = \left(\frac{\mathrm{i}}{2b}\right)^n \phi_{n,c}(x) \tag{23}$$

is said to be the Fourier transform between the p-adic $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$-space and the p-adic $L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$-space.

In this section, we shall always assume that $c = \frac{1}{4b}$.

The main theorem is the following one.

Theorem 3.1. The Fourier transform $\mathcal{F}$ is an isomorphism between the p-adic Hilbert spaces $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$, $L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$ in the sense that it is a linear bijective map from $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ onto $L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$ which preserves the norm and the inner product.

Proof. First, we show that $\mathcal{F}$ sends $f \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ into $\mathcal{F}(f) \in L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$. Write $f = \sum f_n\phi_{n,b}$ and compute $\mathcal{F}(f) = \sum_n g_n\phi_{n,c}$ where $g_n = f_n(\mathrm{i}/2b)^n$; since

$$\sigma_{n,c}^2(g) = \sigma_{n,b}^2(f) \tag{24}$$

we see that $\mathcal{F}(f)$ indeed belongs to $L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$.

Further we have

$$\|\mathcal{F}f\|^2 = \left\| \sum f_n \left(\frac{\mathrm{i}}{2b}\right)^n \phi_{n,c} \right\|^2 = \max_n \left| f_n \left(\frac{\mathrm{i}}{2b}\right)^n \right|^2 \frac{|n!|_p}{|c|_p^n} = \|f\|^2. \tag{25}$$

† In the real case we also have the normalization constant $\sqrt{2\pi}$. It would probably be useful to do the same in the p-adic case. However, at the moment it is not clear what a p-adic analogue of π is.

Similarly, for the scalar product, if $u, v \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$

$$(u, v) = \sum u_n \bar{v}_n \frac{n!}{b^n}. \tag{26}$$

On the other hand, we obtain

$$(\mathcal{F}(u), \mathcal{F}(v)) = \sum_n u_n \left(\frac{\mathrm{i}}{2b}\right)^n \bar{v}_n \left(\frac{-\mathrm{i}}{2b}\right)^n \frac{n!}{c^n} = (u, v). \tag{27}$$

Now we show that $\mathcal{F}$ admits an inverse map $\mathcal{F}^{-1}$. Let $g \in L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$, $g = \sum g_n \phi_{n,c}$. We set

$$\mathcal{G}g = \sum g_n (-2\mathrm{i}b)^n \phi_{n,b}. \tag{28}$$

By the same reasons as above, the map $\mathcal{G}$ sends $g \in L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$ into a map $\mathcal{G}(g) \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ and preserves the norm and the inner product.

Now let $g \in L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$, $g = \sum g_n \phi_{n,c}$, be the Fourier transform of $f \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$. We have:

$$\mathcal{G}(g) = \sum f_n \left(\frac{\mathrm{i}}{2b}\right)^n (-2\mathrm{i}b)^n \phi_{n,b} = f.$$

This proves that $\mathcal{G} = \mathcal{F}^{-1}$. □

Now we wish to relate our definition of the Fourier transform to a more usual integral form. However, to obtain the integral representation for the Fourier transform, we have to extend the p-adic Gaussian (and consequently Lebesgue) integrals. We start from the Gaussian integral. In [10] the Gaussian integral was defined for the following class of analytic functions.

Denote the space of analytic functions $f : U_r \to \mathbb{Q}_p$ by the symbol $\mathcal{A}(U_r)$. Let $b \in \mathbb{Q}_p, b \neq 0$. Then the Gaussian integral

$$\int f(x)\nu_b(\mathrm{d}x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}} f(0) \int x^{2n} \nu_b(\mathrm{d}x) = \sum_{n=0}^{\infty} \frac{b^n}{n!2^n} \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}} f(0)$$

is well defined for each $f \in \mathcal{A}(U_\tau)$, $\tau > \theta_b$ where the constant θ_b is defined by (18). Set $I_a(\mathbb{Q}_p, \nu_b) = \cup_{\tau > \theta_b} \mathcal{A}(U_\tau)$. We shall essentially extend this class of functions integrable with respect to the Gaussian distribution ν_b.

Lemma 3.1. Let $b, d \in \mathbb{Q}_p, b, d \neq 0$. Then the function $\mathrm{e}^{-x^2/2d}$ belongs to the class $I_a(\mathbb{Q}_p, \nu_b)$ iff $|b|_p < |d|_p$.

Theorem 3.2. Let $f \in I_a(\mathbb{Q}_p, \nu_b)$ and $|b|_p < |d|_p$. Then

$$\int f(x)\mathrm{e}^{-x^2/2d} \nu_b(\mathrm{d}x) = \sqrt{t/b} \int f(x)\nu_t(\mathrm{d}x) \qquad t \equiv t_{b,d} = bd/(b+d). \tag{29}$$

Proof. First we prove that the Gaussian integrals in (29) are well defined. As $f(x)$ and $\mathrm{e}^{-x^2/2d}$ belong to the class $I_a(\mathbb{Q}_p, \nu_b)$, these functions belong to the class $\mathcal{A}(U_\tau)$ for some $\tau > \theta_b$. However, $\mathcal{A}(U_\tau)$ is an algebra with respect to the usual multiplication of functions. Thus, the Gaussian integral on the left-hand side of (29) is defined. Further we have $t = b/(1 + b/d)$; but $|1 + b/d|_p = 1$, i.e. $|t|_p = |b|_p$. Hence $\theta_b = \theta_t$ and consequently $I_a(\mathbb{Q}_p, \nu_t) = I_a(\mathbb{Q}_p, \nu_b)$, i.e. the Gaussian integral on the right-hand side of (29) is defined. As both Gaussian integrals are defined by the term integration of power series, it is sufficient to consider the case of monomial functions $f(x) = x^n$, $n = 0, 1, 2, \ldots$. We illustrate the proof by considering the case $f \equiv 1$:

$$A = \int \mathrm{e}^{-x^2/2d} \nu_b(\mathrm{d}x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)! b^k}{(2d)^k (k!)^2 2^k}.$$

On the other hand, we have:

$$\sqrt{t/b} = (1 + b/d)^{-1/2} = \sum_{k=0}^{\infty} (b/d)^k \frac{(-1)^k (2k)!}{(k!)^2 2^{2k}} = A.$$

There we have used the binomial expansion in $\mathbb{Q}_p$ (in fact it is important that $p \neq 2$, i.e. $|\frac{1}{2}|_p = 1$, see [17, 18]). In the general case, $f(x) = x^n$, the proof is little bit more complicated but it is also based on the elementary calculations with the power series.

We now use equality (29) to extend the p-adic Gaussian integral. Let $b, d \neq 0$, $b \neq -d$; let $f \in I_a(\mathbb{Q}_p, \nu_t)$, $t = t_{b,d}$. Then the integral of the function $g_d(x) = f(x)\mathrm{e}^{-x^2/2d}$ with respect to the Gaussian distribution ν_b is defined by equality (29).

On the basis of this generalized Gaussian integral we extend the p-adic Lebesgue integral (by (21)).

Theorem 3.3. Let $|x|_p < \theta_b$, $|y|_p < \theta_c$ (here θ_b, θ_c denote the usual parameter θ with respect to the covariances b, c respectively), then the following relation holds:

$$\mathcal{F}(\phi_{n,b})(y) = \int \phi_{n,b}(x)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x. \tag{30}$$

Proof. We have:

$$\begin{aligned}
\int \phi_{n,b}(x)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x &= b^{\frac{-1}{4}} \int H_{n,b}(x)\mathrm{e}^{\frac{-x^2}{4b}}\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x = b^{\frac{1}{4}} \int \mathrm{e}^{\frac{x^2}{4b}+\mathrm{i}xy} H_{n,b}(x)\nu_b(\mathrm{d}x) \\
&= (-1)^n b^{\frac{1}{4}} \int \mathrm{e}^{\frac{x^2}{4b}+\mathrm{i}xy} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \nu_b(\mathrm{d}x) = b^{\frac{1}{4}} \int \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{\frac{x^2}{4b}+\mathrm{i}xy})\nu_b(\mathrm{d}x) \\
&= b^{\frac{1}{4}}\mathrm{e}^{by^2} \int \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{\frac{1}{4b}(x+2\mathrm{i}by)^2}\nu_b(\mathrm{d}x) \\
&= \left(\frac{1}{2b\mathrm{i}}\right)^n b^{\frac{1}{4}}\mathrm{e}^{by^2} \frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(\mathrm{e}^{-by^2} \int \mathrm{e}^{\frac{x^2}{4b}+\mathrm{i}xy})\nu_b(\mathrm{d}x)\right) \\
&= \left(\frac{1}{2b\mathrm{i}}\right)^n b^{\frac{1}{4}} 2^{\frac{1}{2}}\mathrm{e}^{by^2} \frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(\mathrm{e}^{-by^2} \int \mathrm{e}^{\mathrm{i}xy}\nu_{2b}(\mathrm{d}x)\right) \\
&= \left(\frac{1}{2b\mathrm{i}}\right)^n b^{\frac{1}{4}} 2^{\frac{1}{2}}\mathrm{e}^{by^2} \frac{\mathrm{d}^n}{\mathrm{d}y^n}\mathrm{e}^{-2by^2} = \left(\frac{\mathrm{i}}{2b}\right)^n (-1)^n c^{\frac{-1}{4}}\mathrm{e}^{\frac{y^2}{4c}} \frac{\mathrm{d}^n}{\mathrm{d}y^n}\mathrm{e}^{-\frac{y^2}{2c}} \\
&= \left(\frac{\mathrm{i}}{2b}\right)^n \phi_{n,c}(x)
\end{aligned}$$

where, as usual, $b = 1/4c$. We have used the generalized Gaussian integral for $d = -2b$, i.e. $t_{b,d} = 2b$. □

Now we wish to prove the main properties of the Fourier transform.

Theorem 3.4. Assume that all the variables are in their convergence regions: we have

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathcal{F}f = \mathrm{i}^k\mathcal{F}(y^k f) \qquad \mathcal{F}\frac{\mathrm{d}^k}{\mathrm{d}x^k} f = (-\mathrm{i}y)^k \mathcal{F}f. \tag{31}$$

Proof. First we remark that all L_2-functions are infinitely differential and their derivatives belong to the same L_2-space (see [9]).

We only prove the first statement for $k = 1$, the other statements being obtained in a similar way. We observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathcal{F}f)(x) = \sum f_n \left(\frac{\mathrm{i}}{2b}\right)^n \frac{\mathrm{d}}{\mathrm{d}x}\phi_{n,c}(x) = -\frac{1}{2}\sum f_n \left(\frac{\mathrm{i}}{2b}\right)^n \left[\phi_{n+1,c}(x) - \frac{n}{c}\phi_{n-1,c}(x)\right]. \tag{32}$$

On the other hand, we compute

$$\begin{aligned}\mathcal{F}(yf)(y) &= \mathcal{F}\sum f_n y\phi_{n,b}(y) = \sum f_n \mathcal{F}[b\phi_{n+1,b}(y) + n\phi_{n-1,b}(x)]\\ &= \mathrm{i}/2\sum f_n \left(\frac{\mathrm{i}}{2b}\right)^n [\phi_{n+1,c}(x) - \frac{n}{c}\phi_{n-1,c}(x)]\end{aligned} \tag{33}$$

which proves (31) (in the case $k = 1$).

Formally we define the convolution of two maps f, g as

$$(f \star g)(z) = \int f(x)g(z-x)\,\mathrm{d}x \tag{34}$$

(also here we suppose $|x|_p, |z|_p < \theta$). However, in general the convolution of two L_2-functions does not belong to the L_2-space. In fact, there the situation is more 'pathological' than in the real case. By direct computations we can prove that the convolution of two Hermitian functions, $f = \phi_{0,b}$ and $g = \phi_{1,b}$, does not belong to the L_2-space. Therefore, it is not clear on what domain the standard equality: $\mathcal{F}(f \star g) = \mathcal{F}f\ \mathcal{F}g$ can be considered to be valid. □

4. The spectrum of the momentum operator

In this section we study the momentum representation by using the properties of the Fourier transform. We consider the case of a particle moving in one dimension (the extension to the d-dimensional case being immediate).

If $f \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ we define the momentum operator by

$$\hat{\boldsymbol{p}}f(x) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}f(x). \tag{35}$$

Now applying the Fourier transform we have

$$\mathcal{F}\hat{\boldsymbol{p}}f(x) = x\mathcal{F}f = \hat{\boldsymbol{x}}\mathcal{F}f$$

so, we obtain that the spectrum of the momentum operator in $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ coincides with spectrum of the position operator in $L_2^{(c)}(\mathbb{Q}_p, \mathrm{d}x)$, where, as before, $c = 1/4b$.

In [10] we studied the spectral properties of the position operator in the space $L_2(\mathbb{Q}_p, \nu_b)$; since the spaces $L_2(\mathbb{Q}_p, \nu_b)$ and $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ are isomorphic, we obtain the following theorem.

Theorem 4.1. Let $|\lambda|_p < \theta_c$. Then λ belongs to the spectrum of the momentum operator $\hat{\boldsymbol{p}}$. The point spectrum of the momentum operator $\hat{\boldsymbol{p}} : L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x) \to L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$ is the empty set.

5. Exactness of the measurement for the position and momentum of a harmonic oscillator

In this section we briefly consider the Schrödinger equation for a harmonic oscillator on $\mathbb{Q}_p$, i.e.

$$-\frac{\overline{h}^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x) \qquad x \in \mathbb{Q}_p \tag{36}$$

with $\psi \in L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$, $\overline{h}, m, \omega, E \in \mathbb{Q}_p$.

Remark. In our approach we consider the Planck constant $\overline{h}$, the frequency ω and the mass m of the harmonic oscillator as measured with a finite exactness. Thus, in equation (36) the numbers $\overline{h}, \omega, m$ have the form

$$a = a_{-n}p^{-n} + \cdots + a_0 + \cdots + a_m p^m \tag{37}$$

with the digits $a_j = 0, 1, \ldots, p-1$. Here an exactness $\Delta(a) = 1/|a|_p = 1/p^n$ is fixed for each quantity $\overline{h}, \omega, m$. The same is valid for the energy levels E which also have the form (37).

There are solutions of (36) (in $L_2^{(b)}(\mathbb{Q}_p, \mathrm{d}x)$) of the form:

$$\psi(x) = \phi_{n,b}(x) \qquad \text{where } b = \frac{\overline{h}}{2m\omega} \tag{38}$$

and $E = \overline{h}\omega(n + \frac{1}{2})$ for any natural n.

The situation is similar to the case of the ordinary quantum mechanical harmonic oscillator (over $\mathbb{R}$ with $\mathbb{C}$-valued functions), except for the fact that the wavefunction $\psi(x)$ is defined only for x s.t. $|x|_p < \theta$. If we follow the physical interpretation of the p-adic norm [10] of a physical quantity A as the inverse of the exactness of its measurement, i.e.

$$\delta(A) = \frac{1}{|A|_p} \tag{39}$$

then, the restriction on the domain of the wavefunction expresses the limitation of the exactness of the measurement of the coordinate of a quantum particle: $\delta(x) \geqslant 1/\theta_b$ (we remark that this definition of $\delta(x)$ is not directly related to the definition of uncertainty in ordinary quantum mechanics). Thus, in this interpretation it is not possible to measure the position of a harmonic oscillator with an exactness which is better than $1/\theta_b$:

$$\delta(x) \geqslant \frac{1}{\theta_b} = p^{\frac{1}{2(p-1)}}\sqrt{\frac{|2m\omega|_p}{|\overline{h}|_p}}. \tag{40}$$

In a similar way, by performing a p-adic Fourier transform, we obtain that we cannot measure the momentum of the harmonic oscillator with an exactness which is better than $1/\theta_c$:

$$\delta(\boldsymbol{p}) \geqslant \frac{1}{\theta_c} = p^{\frac{1}{2(p-1)}}\frac{1}{\sqrt{|2m\omega|_p}\sqrt{|\overline{h}|_p}}. \tag{41}$$

We note that the factor $\mathrm{e}^{\frac{\mathrm{i}xy}{\overline{h}}}$ has been used instead of $\mathrm{e}^{\mathrm{i}xy}$ in the integral representation (30) of the Fourier transform in order to take into account the dimension of physical quantities.

In this interpretation we obtain that not only is there a minimum value for the product of the precision of the measurements of position and momentum, but also, that there is a minimal value to the precision with which we can measure the *single* quantity position or momentum. If we interpret $\delta(\overline{h}) = \frac{1}{\sqrt{|\overline{h}|_p}}$ as 'an exactness of measurement' for $\overline{h}$, then by (40) and (41) we have:

$$\delta(x)\delta(p) \geqslant p^{\frac{1}{p-1}}\delta(\overline{h}). \tag{42}$$

These 'inexactness relations' obviously have some relation, but are not equivalent, with the Heisenberg uncertainty relations.

Finally we make two remarks. First, the term $p^{\frac{1}{p-1}}$ approaches 1 when $p \to \infty$. Hence, we obtain the inexactness relations

$$\delta(x)\delta(p) \geqslant \delta(\overline{h}) \tag{43}$$

in the limit $p \to \infty$. We also note that $\frac{1}{\sqrt{|\overline{h}|_p}}$ can be very small as compared with $\overline{h}$.

6. The p-adic description of the finite exactness of the measurement

The formalism of p-adic Hilbert space quantization and its corresponding physical interpretation is described in [20]. Here we briefly summarize some essential parts of this formalism.

We always use real numbers to describe measurement procedures both in classical and quantum physics. This 'real description' has been used for a long time (at least since Galilei's work). We are so used to this description that we are no longer surprised that in this description we operate with physical quantities as if they could be measured with an infinite exactness. A real number having an infinite number of decimal digits, the formalism implies all these digits might be measured, at least in principle. However, from the practical point of view every concrete experiment permits only a finite exactness of measurement.

Is it possible to include this finite exactness in a mathematical formalism? We shall see how our p-adic formalism can be interpreted in this sense.

What can we obtain in a measurement $\mathcal{S}$? Let us choose the unit of a measurement to be 1 and let us fix a natural number m (corresponding to the scale of this measurement). As results of $\mathcal{S}$ we can obtain only quantities of the form

$$x = \pm\left(\frac{x_{-k}}{m^k} + \cdots + \frac{x_{-1}}{m} + x_0 + \cdots + x_s m^s\right) \tag{44}$$

where $x_j = 0, 1, \ldots, m-1$ are digits in our scale. Denote the set of all such x by $\mathbb{Q}_{m,\mathrm{fin}}$.

It is natural to take $\frac{1}{m^k}$ as a measurement of the 'exactness' of $\mathcal{S}$. Since we can practically only achieve a finite exactness in $\mathcal{S}$ which we express by saying that there exists a fixed number $k = k(\mathcal{S})$ such that the maximal achievable exactness of $\mathcal{S}$ is equal to $\delta(\mathcal{S}) = 1/m^k$. This means that we can be sure only in the digit x_{-k} but the next digit $x_{-(k+1)}$ is not well defined in $\mathcal{S}$ (in this fixed scale).

We wish to create a number system which describes only finite exactness of measurements. The set of 'physical numbers' $Q_{m,\mathrm{fin}}$ will be taken to be the basis of our considerations.

First we are interested in the construction of the field of real numbers R on the basis of $Q_{m,\mathrm{fin}}$. The field R is the completion of $Q_{m,\mathrm{fin}}$ with respect to the real metric $\rho(x, y) = |x - y|$ corresponding to the usual absolute value (valuation) $|\cdot|$. This metric describes absolute values of physical quantities (with respect to the fixed coordinate system). We define on $Q_{m,\mathrm{fin}}$ a new valuation corresponding to the exactness $\delta(\mathcal{S})$.

Set $|x|_m = m^k$ for x given by equation (44) (we assume that $x_{-k} \neq 0$); $|\cdot|_m$ is a pseudovaluation, in the sense that it has the following properties:

- $|x|_m \geqslant 0$ and $|x|_m = 0$ iff $x = 0$;

- $|xy|_m \leqslant |x|_m |y|_m$;
- $|x+y|_m \leqslant \max(|x|_m, |y|_m)$ (the strong triangle inequality).

Set $\rho_m(x, y) = |x-y|_m$ and complete $\mathbb{Q}_{m,\mathrm{fin}}$ with respect to this metric†. Denote this complete metric space by $\mathbb{Q}_m$ (m-adic numbers). This is a ring with respect to extensions of the usual operations of addition and multiplication. If $m = p$ is a prime number, then we obtain the field of p-adic numbers $\mathbb{Q}_p$.

Of course, from the mathematical point of view it would be better to work in a field than in a ring. Therefore, in this paper we have studied quantum field models over $\mathbb{Q}_p$. However, from the physical point of view it is better to use the general scheme on the basis of m-adic numbers.

As in the p-adic case, for any $x \in \mathbb{Q}_m$ we have a unique canonical expansion (convergent in the $|\cdot|_m$-norm) of the form

$$x = x_{-n}/m^n + \cdots x_0 + \cdots + x_k m^k + \cdots \tag{45}$$

where $x_j = 0, 1, \ldots, m-1$, are the 'digits' of the m-adic expansion. This expansion contains only a finite number of digits corresponding to negative powers of m. We interpret these numbers as providing a description of the finite exactness of a measurement. However, expansion (45) shows that there is a new element in the m-adic description which is not present in the description of real numbers. There exists quantities described by (45) with an infinite number of digits corresponding to positive powers of m. It is very natural to consider such quantities as infinite quantities (with respect to our fixed unit 1). At the moment we are not sure whether such quantities might be useful in physics. However, there is always the possibility of restricting our attention to finite quantities, i.e. x described by (45) with $x_l = 0$ for all l larger that some finite k (depending on x).

Now the difference between real and m-adic descriptions of measurement is clear. If the exactness is infinite and the values of all observables are finite, then we have the real numbers description. If the exactness is finite and some values of observables may be infinite, then we have the m-adic numbers description.

Here m plays the role of a parameter characterizing the structure of the fixed scale. Different scales are useful for different experiments. However, different m-adic descriptions are equivalent from a practical point of view, for example the exactness $\delta(\mathcal{S}) = 1/2^k$ can be realized approximately as $\delta'(\mathcal{S}) = 1/3^l$ for some suitable l. But at the same time the rings $\mathbb{Q}_m$ and $\mathbb{Q}_{m'}$, $m \neq m'$, are not isomorphic.

Now we consider the process of quantization. Further, we shall follow the so-called statistical interpretation of quantum states.

Any measurement process has two steps:

(1) a preparation procedure $\mathcal{E}$;

(2) a measurement of a physical quantity B_{phys} in the states which were prepared with the aid of $\mathcal{E}$.

According to the statistical interpretation, a quantum state represents an ensemble of similarly prepared (with the aid of $\mathcal{E}$) systems. Typically a preparation procedure $\mathcal{E}$ is based on the filtration with respect to one of properties of a physical object. In particular, a wavefunction describes the statistical distribution of this fixed property.

Let A_{phys} be the fixed physical quantity and a preparation procedure $\mathcal{E} = \mathcal{E}_A$ is realized as the filter with respect to values of A_{phys}. $\mathcal{E}_A$ generates statistical ensembles of states ψ_α corresponding to values α of A_{phys}. Further, using ψ_α, we can prepare mixed states where ψ_α arises with a probability P_α. It is important to note that $\psi_\alpha = \psi_\alpha(\mathcal{E}_A)$.

† This is the so-called ultrametric, i.e. the strong triangle inequality $r(x, y) \leqslant \max(r(x, z), r(z, y))$ holds.

Now, consider the second step of a measurement process. There is the state $\psi = \psi(\mathcal{E}_A)$ prepared by the preparation procedure $\mathcal{E}_A$ and a physical quantity $B_{\rm phys}$. Values of $B_{\rm phys}$ are measured on the statistical ensemble ψ.

The standard mathematical description of a measurement process uses as the space of states a complex Hilbert space $\mathcal{H}$. States are normalized vectors $\psi \in \mathcal{H}$, physical observables $B_{\rm phys}$ are described by self-adjoint operators in $\mathcal{H}$. The expansion of ψ with respect to an orthogonal basis $\{e_n\}$:

$$\psi = \sum c_n e_n \tag{46}$$

has a standard statistical interpretation (see, e.g. [13, 14]).

The dependence $\psi = \psi(\mathcal{E}_A)$ does not play any role in this $\mathcal{H}$-description, in the following sense. Suppose the state ψ is constructed with the aid of $A_{\rm phys}$, $\psi = \psi_A = \sum c_{nA} e_{nA}$ (here the states e_{nA} correspond to the values λ_n of $A_{\rm phys}$). Then in principle ψ can be constructed with the aid of another physical quantity $A'_{\rm phys}$, $\psi = \psi'_A = \sum c_{nA'} e_{nA'}$. As ψ_A and $\psi_{A'}$ coincide as the elements of $\mathcal{H}$, all physical properties of ψ_A and $\psi_{A'}$ coincide.

However, it is not evident that if we measure the quantity $B_{\rm phys}$ in the state ψ_A we obtain the same result as in the measurement on the state $\psi_{A'}$, despite the equality

$$\psi = \psi_A = \sum c_{nA} e_{nA} = \sum c_{nA'} e_{nA'} = \psi_{A'} \tag{47}$$

in the complex Hilbert space (because this equality is a mathematical equality; in practice it is realized as an approximation of one finite mixture by another (also finite) mixture; this approximation may change the statistical behaviour crucially).

We shall now propose a formalism of quantization in a p-adic Hilbert space $\mathcal{H}_p$. This formalism differs from the complex Hilbert space formalism. The main difference is connected with the question about the equivalence of different representations of a ψ-function. The mathematical structure of the $\mathcal{H}_p$-formalism automatically generates non-equivalent representations of a ψ-function. Here the dependence $\psi = \psi(\mathcal{E}_A)$ is essential. Equality (47) does not imply that all physical properties of two statistical ensembles ψ_A and $\psi_{A'}$ coincide.

In the p-adic Hilbert space formalism we have to consider the exactness of a measurement of values of $A_{\rm phys}$. This exactness generates a metric on the space of states. Following our general scheme (with the mathematical restriction, $m = p$ is a prime number) we choose the unit 1 and the prime number p which describes the scale of a measurement process. Let $\Lambda = \{\lambda\}$ be values of a physical quantity $A_{\rm phys}$. Suppose that these values are measured with the exactness $\delta(A_{\rm phys}) = 1/p^k$, $k \in \mathbb{Z}$, i.e.

$$\lambda = \lambda_{-k}/p^k + \cdots + \lambda_s p^s \qquad \lambda_j = 0, 1, \ldots, p-1. \tag{48}$$

This is an expression of the fact that we are sure of the digit λ_{-k} but not sure of the next digit $\lambda_{-(k+1)}$. Using a preparation procedure $\mathcal{E} = \mathcal{E}(A_{\rm phys}, \Lambda)$, we construct the states corresponding to the values $A_{\rm phys} = \lambda$. Denote these states by the symbols $\psi_\lambda, \lambda \in \Lambda$. Then it is possible to prepare statistical mixtures of these states. They are, by definition, linear combinations of the ψ_λ:

$$\psi = \sum_{n=1}^{m} c_n e_{\lambda_n} \qquad c_n = a_n + \mathrm{i} b_n \qquad a_n, b_n \in Q. \tag{49}$$

The space $\mathcal{H}_{\rm fin}$ of vectors (49) is complete with respect to a metric corresponding to the exactness of a measurement of values of $A_{\rm phys}$. This completion is a p-adic Hilbert space $\mathcal{H}_p$. The main difference with the complex case is that the metric (consequently, the completion) depends on A_{phys}, thus $\mathcal{H}_p$ depends on A_{phys}. As in the complex formalism, physical quantities B_{phys} are realized by linear operators B in $\mathcal{H}_p$. However, $\mathcal{H}_p$ was built on the basis of the finite exactness of a measurement of A_{phys}. It would be very naive to hope that the $\mathcal{H}_p$-formalism could predict exact values of B_{phys}. We are able to predict values of B_{phys} only with a fixed exactness $\delta(B)$ which is connected with the exactness of a measurement of $A_{\mathrm{phys}}, \delta(A)$. We suppose that the values of an arbitrary physical quantity B_{phys} could be predicted only with the exactness $\delta(B) = 1/\|B\|$, where $\|B\|$ being the operator norm of B, if $\|B\| \neq 0$ (the dependence on A is due to the dependence of the p-adic Hilbert space $\mathcal{H}_p$ on the preparation procedure).

Thus, the main difference between the standard quantum formalism and the p-adic formalism can be seen in the fact that the latter takes into account the finite exactness of measurements of physical quantities. In our p-adic approach one has the possibility of predicting the exactness of measurement of a physical quantity B_{phys} which is described by an operator B in the p-adic Hilbert space (as the inverse to the norm of B). This already applies in the case where λ is an eigenvalue of $B, Be = \lambda e$, i.e. in the p-adic approach we cannot say that in the state e the physical quantity has exactly the value λ (with the probability 1); this value also can be measured only with the exactness $\delta(B) = 1/\|B\|$. For example, the p-adic position and momentum can be interpreted as *usual position and momentum, but measured with a finite exactness*. In this sense our formalism contains a lesser idealization than the usual quantization based on complex numbers. Of course, as in any mathematical formalism describing a physical model, our theory also contains some ideal elements for which seem to be impossible to verify in an experiment. These are infinite quantities which are described by p-adic numbers having an infinite number of digits in the canonical expansions.

The idea of using a finite exactness of measurements to modify the present quantum formalism has been expressed before, for example by Prugovecki [21–23] which contains an analysis of the role of exactness of individual measurements in quantum theory. However, Prugovecki tried to describe a finite exactness of measurements by real numbers, giving the main attention to a finite exactness of measurement for incompatible observables described by non-commuting operators. From our point of view this problem is already very important in the case of a single physical quantity. Another approach based on a finite exactness of measurement was developed in the framework of so-called positively defined operator-valued measures (see Davies [24], Ludwig [25], Holevo [26]).

The probability interpretation of quantum states in the p-adic quantum formalism is similar to the standard one. However, there are some differences concerning the presence in a p-adic Hilbert space of quantum states which do not have the standard probability interpretation in the framework of the Kolmogorov axiomatics [27].

We start from the consideration of quantum states which permit a standard statistical interpretation. Suppose that the physical quantity B_{phys} is described by the symmetric operator B in the p-adic Hilbert space. Let e_n be the system of eigenvectors of $B : Be_n = \lambda_n e_n$. We consider the finite mixture of this vectors with rational coefficients:

$$\psi = \sum_{k=1}^{N} c_k e_k \qquad c_k \in \mathbb{Q} \tag{50}$$

satisfying to the normalization condition $\sum_{k=1}^{N} c_k^2 = 1$. Then, as in the usual quantum formalism, we predict that the physical quantity B_{phys} yields in the quantum state ψ the value λ_m with probability c_k^2. The same probability interpretation can be used for mixtures with coefficients $c_k = a_k + \mathrm{i}b_k$, $a_k, b_k \in \mathbb{Q}$. In fact, this is a large class of quantum states (it might even be reasonable to assume that all quantum states, which may be prepared by some physical preparation procedure, have the form (50)).

However, a p-adic Hilbert space also contains quantum states which do not permit the standard statistical interpretation. In particular, these are finite mixtures of eigenvectors with non-rational p-adic coefficients. We propose using, for such states, the probability interpretation based on the so-called p-adic probability theory, see [1, 3, 28, 29] (which is based on a generalization of the frequency approach of Mises [30, 31], corrected by further developments of Kolmogorov, Martin-Lof, Chaitin and others).

In this probability formalism probabilities are defined as limits of relative frequencies, but these limits are considered with respect to the p-adic topology on the field of rational numbers $\mathbb{Q}$. It is evident that all relative frequencies are rational numbers; thus their asymptotics can be studied, not only in the standard real metric on $\mathbb{Q}$, but also in any metric (or more generally topology) on $\mathbb{Q}$ and, in particular, in one of the p-adic topologies. In [1, 3, 28, 29] we presented an extended class of p-adic random sequences which did not have the property of the statistical stabilization with respect to the real metric (the relative frequencies oscillate with respect to the real metric, but they stabilize with respect to the p-adic metric). Further, p-adic frequency probability theory was reformulated by using the measure-theoretical approach [28, 29] where p-adic probabilities were defined as normalized $\mathbb{Q}_p$-valued measures. The properties of p-adic frequency probabilities were used to formulate the corresponding system of axioms. One of the interesting properties of p-adic probabilities is the possibility of realizing negative rational numbers as p-adic probabilities [28, 29].

Therefore our p-adic Hilbert space quantum formalism implies the existence of generalized quantum states with unusual statistical behaviour. This statistical behaviour is totally chaotic from the point of view of the standard probability theory, although it is well described by p-adic probability distributions (see [32–35]). The crucial point of these investigations is the possibility to consider negative probability distributions as well defined mathematical objects in the framework of the p-adic probability theory. In particular, this gives us the possibility to propose a p-adic probability description of the Einstein–Podolsky–Rosen paradox (see [36, 37]).

We can also interpret the use of p-adic (or m-adic) numbers as a way of inducing a fundamental length in quantum physics theory. Let us consider a p-adic space-time model:

$$M_4 = U_{1/l_x} \times U_{1/l_y} \times U_{1/l_z} \times U_{1/l_t}$$

where l_x, l_y, l_z, l_t have the form p^k, $k = 0, \pm 1, \ldots$. In this discrete space-time l_x, l_y, l_z, l_t are the minimal lengths for space and time intervals. It is evident that the p-adic space-time M_4 is an additive group. We can define the analogue of Lorentz transformations in M_4. We plan to study this model (in particular, the Lorentz invariance of p-adic quantum fields model) in subsequent papers.

We have presented (one of the possible) interpretations of p-adic quantum models. What about the classical limit of p-adic quantum models? We expect that by using the calculus of pseudodifferential operators [1] we can study the relations between quantum and classical models on p-adics.

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